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Tandem conceptual progression (TCP): Pilot case study of a cognitively diverse student's use of adjacent problem-solving and answer-checking schemes

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ABSTRACT

Children experiencing difficulties learning mathematics often have a long-embedded coping mechanism of looking to others as authorities for the correctness of their solutions. In this *pilot* case study, we demonstrate ways in which promoting their checking of their own answers can empower their development. Specifically, we examine answer-checking schemes that a cognitively diverse 6th grader with difficulties learning mathematics used when solving additive tasks. We draw on constructivist scheme theory as a framework to analyze data from a year-long teaching experiment, demonstrating a rather rapid progress in his problem-solving schemes, from counting-all to break-apart-make-ten. Along with this rapid conceptual progress, we found that his answer-checking schemes developed in tandem with the problem-solving schemes, typically being one cognitive step behind the latter, that is, advancing from no answer-checking to a numerical count-on scheme. During problem-solving, he may have used schemes at either a participatory or anticipatory stage, whereas for answer-checking he mostly used schemes at the anticipatory stage. We discuss theoretical and practical implications of these novel findings about numerical progress in a cognitively diverse student.

1. Introduction

1.1. Answer-checking

With much credit to Pólya (1957), the ways in which learners check answers they obtain during mathematical problem solving have garnered mathematics educators' attention for decades. He stressed that the final phase of mathematical problem-solving is to

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look back and evaluate the solution, providing an opportunity to make new connections between prior knowledge and the methods used to solve the problem (or task). This is an example of a type of reflection needed for accommodation, and thus learning, to happen (Steffe & Cobb, 1988). A constructive step in this phase is answer-checking—determining if the checked answer fits with the initial solution. As Pólya and others (Kilpatrick, 1987; Schoenfeld, 1987) have emphasized, answer-checking is one metacognitive tool that children can use to prompt them to reflect on the outcomes of their mathematical activity.

Yet, many children experiencing difficulties in learning mathematics look to others as authorities for the correctness of their answers (Brousseau & Warfield, 1999). Thus, they rarely check their own answers or reflect on their ways of reasoning. This lack of reflection seems to diminish an important source for these children's ability to learn and make progress in their mathematical reasoning (see Theoretical Framework).

Despite the vital role that answer-checking may play in reflection, and thus learning, as a conceptual change, this is an area rarely studied for children with difficulties learning mathematics. Thus, we studied schemes that a cognitively diverse student used to solve arithmetical problems (hereafter referred to also as tasks) and how they related to his answer checking strategies. Specifically, we addressed the research question: What might be the conceptual relationships between a student's answer-checking schemes and their problem-solving schemes?

2. Theoretical framework

We first present general constructs of a constructivist stance that underlies our study. We link those to two non-constructivist research programs that seem relevant for further situating our work. Then, we present content-specific constructs (schemes) pertaining to additive operations on whole numbers. As a heads-up for the reader, in the first section we present a slightly reconceptualized, and we believe clearer model, of the interrelationships among three key constructs: scheme, strategy, and conception.

2.1. General constructs of our constructivist stance

The general constructs in our study draw on Piaget's (1964) constructivist stance. This stance postulates that cognitive functioning, like any living system, is afforded and constrained by structures ('apparatuses') that are being continually adapted. Briefly, cognition functions through restoration of equilibrium that is enabled by schemes – the apparatus underlying the complementary mechanisms of assimilation and accommodation. Henceforth, we shall refer to the latter by reorganization (see Steffe, 2010; Tzur, 2019b). Glasersfeld (1995) elaborated on this stance with what is known as *scheme theory*, which we explain shortly.

Piaget (1964), in refuting the behaviorist notion of stimulus-response, stressed that any cognitive experience begins with assimilation into available schemes. Assimilation into available schemes takes place when one is processing "external input" and/or from intra-actions within their mental system. He asserted:

A stimulus is really a stimulus only when it is assimilated into a structure and it is this structure which sets off the response. Consequently, it is not an exaggeration to say that the response is there first, or if you wish, at the beginning there is structure (p. 15).

Glasersfeld (1995) elaborated on the notion of scheme, postulating a single, three-part "mental apparatus" serving as a building block of cognitive processes, which affords and constrains assimilation, as well as cognitive reorganization. This single structure

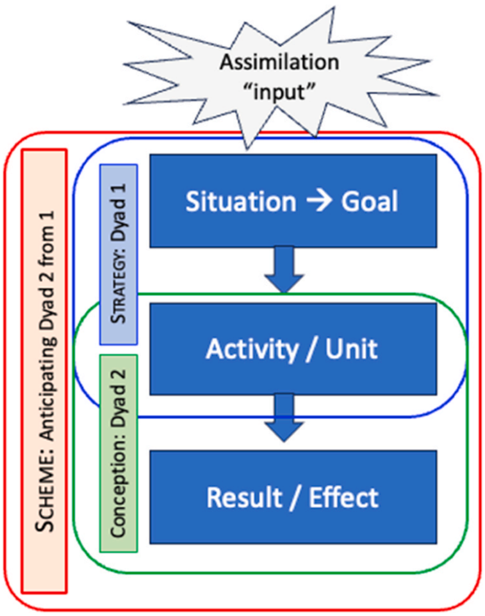

Fig. 1. Strategy and conception as two dyads comprising a three-part scheme.

comprises three sequentially linked parts: (a) a recognition template, termed *situation,* which incorporates "input" into previously used, repeatable patterns of action and gives rise to a *goal*, (b) a mental *activity* directed toward that goal, and (c) a *result* that may be anticipated prior to acting or noticed afterward, in hindsight.

Tzur and Simon (2004) distinguished these two types of activity outcomes, calling the one anticipated in advance *'result*' and the one linked in anticipation after using the activity *'effect'*. They thus coined *conception* the notion activity-effect relationship as a dyad within the three-part notion of scheme. Here, we further accentuate the prior (first) dyad, which comprises the anticipated link between a situation/goal and the activity on some unit items. We refer to this other dyad as *strategy*, as it fits with this construct's dictionary definition: "Devising or employing plans [or stratagems] toward a goal" (Merriam-Webster, n.d.-a). Fig. 1 depicts the three possible linkages among the constructs, strategy (first dyad), conception (second dyad), and scheme as a structure linking both dyads in anticipation.

Let us illustrate this with the counting-on scheme (Steffe et al., 1983; Steffe & Cobb, 1988; Steffe & Glasersfeld, 1985). A child may be asked to walk seven steps over floor tiles and mark this last tile "7." Then, the child walks 5 more steps and marks that last tile "5." The teacher, standing on tile #7 to block the first seven tiles from the child's sight, asks: How far are you from the start? If assimilating this task into a counting-on scheme, the child recognizes two, already counted, numerical collections and sets the goal to determine the yet-to-be-definite numerical collection that would integrate (unite) both collections into a single number. This may trigger an activity of counting with the unit items operated on being one's own counting acts ("double-count"): 7; 8-is-1, 9-is-2, … 12-is-5. In such a case, we attribute to the child the use of a count-on strategy. Then, our model of their reasoning producing that activity leads us to consider the child has anticipated that, once ceasing the double-counting activity, they would be able to state the *number* of steps that could have been taken from the start. We thus attribute to the child the abstraction of a conception (situation-activity dyad) of count-on for an anticipated total of ones in an integrated, numerical collection. In this example, and critical for our study, once the teacher stops blocking the sight of all tiles, the child could also use a count-all scheme, counting from 1 all the way to 12 (while noting that the first stop still is at "7").

An observer can create, from noticeable behaviors (e.g., actions, language) of the child a model of the goal-directed activities that give rise to schemes while they are engaged in solving tasks. If the effect of a goal-directed activity aligns for the child with the anticipated result, then the situation is assimilated into the previously constructed scheme as another repeatable, similar experience. If the effect is unanticipated it may cause a perturbation and the child may opt for reorganizing the scheme. This may involve change of goal, of activity, and/or of result.

Steffe and Cobb (1988) have theorized that, for teaching to bring about a desired reorganization, the child's available schemes need to be at a level where their interpretation of the situation supports the cognitive change intended by the teacher. In other words, a teacher should have a good idea (model) of the child's available schemes and use tasks that allow the child to use those schemes to willingly engage in the solution processes while still posing some challenge to their extant scheme(s) (see Simon & Tzur, 2004). However, Simon and Tzur (2004) emphasized that it is not enough for the teacher to present a task. Rather, to abstract the intended scheme by reorganizing prior, available scheme(s) the child must both actively engage in solving the task and in reflecting on its outcomes, that is, compare actual effects of the activity to the anticipated results.

Tzur and Simon (2004) postulated that reorganizing a scheme into a more advanced scheme typically occurs through two stages (see Fig. 2). These stages are distinguished based on the extent to which the second dyad (conception) of a newly constructed scheme has been linked with the first dyad (strategy). They termed the first stage the *participatory stage* and the second the *anticipatory stage*. We use the term stage in line with Glasersfeld and Kelley's (1982) twofold notion, involving (a) a stretch of time characterized by something that remains constant (e.g., an activity, a state) that differs qualitatively from the preceding and succeeding stretches of time and (b) are progressing towards an expected end state.

When a scheme is at the participatory stage, it is provisional, in that only the second dyad, a linkage between the activity and its effect, has been abstracted. A linkage of that activity-effect "dyad" to the situation part of the scheme is yet to be constructed (Tzur & Simon, 2004; Simon et al., 2016). Thus, a learner cannot yet independently assimilate a situation and anticipate its novel effect. This lack of independence means that the learner needs some prompting to operate with a newly evolving scheme (Tzur & Lambert, 2011). Otherwise, the child may not recognize the situation, so they are likely to fold-back (Pirie & Kieren, 1994) to a prior, anticipatory scheme. The anticipatory-participatory stage distinction builds on the idea that what a child is constructing at a particular stretch of time and is still participatory builds on and reorganizes what has been available at the preceding anticipatory stages.

Tzur and Simon (2004) termed the second stage the anticipatory stage, as the learner has then abstracted links between the two dyads, and can thus anticipate results from the situation, possibly without running the activity. At this stage, the learner can recognize the situation, set the goal, and bring forth the relevant strategy spontaneously and purposefully (Tzur, 2007). Tzur and Simon (2004) proposed that children may struggle to construct robust schemes because once they have constructed a scheme at the participatory stage they appear as if they "have the concept."

We introduce here the main construct that arose through our analysis, *tandem conceptual progression* (TCP). As this construct arose in hindsight, through our retrospective analysis and not through study design to purposely figure it out, we explicate it upfront. In our data analysis, we shall demonstrate a twofold role that teaching attuned to TCP can serve. Thus, here we discuss it briefly.

We use the TCP construct to stress the coupled nature of conceptually adjacent schemes. We chose the term "tandem" because it connotes "consisting of things or having parts arranged one behind the other," as well as "working or occurring in conjunction with each other" (Merriam-Webster, n.d.-b). TCP thus introduces a novel stance on a student's growth along a conceptual progression, such as the one in additive reasoning on which we focus in this study. Such a progression seems to involve a constant interplay between schemes the learner has constructed at the anticipatory stage that they can use for answer-checking, and next-up schemes that are in the process of being constructed.

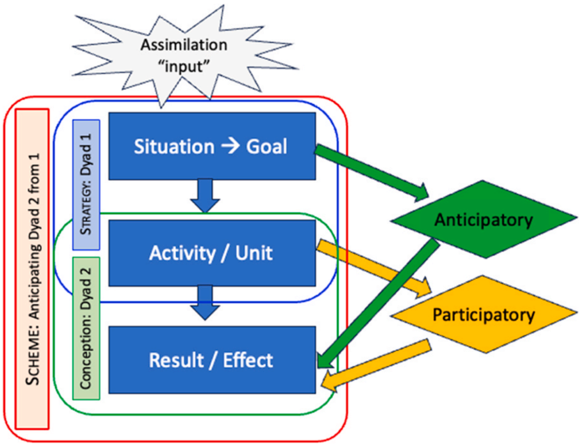

Fig. 2. Two stages in the construction of schemes.

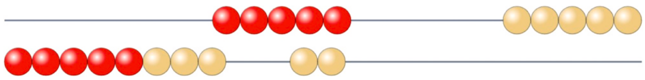

Fig. 3. Using the Rekenrek to model 8 + 7[5]

[5] Throughout we use equations as a simple way to communicate the problems to the reader. When working with MH, all problems were presented verbally or physically, not in an equation format.

When a learner solves a task using an upper-level scheme and then checks the answers using a solidly available, anticipatory scheme that is one (or two) conceptual steps behind, it is most likely they will assimilate the latter back into the former. If both answers match, this assimilation often boosts their confidence and sense of competence in their construction of the upper-level scheme. If the answers differ, it still affords the learner with an opportunity to reorganize the upper-level scheme, so its results agree with the results from the available schemes. In either case, assimilation of a "trusted" scheme that is one step behind into an emerging scheme can promote learning.

As our case study shows, being attuned to TCP seemed to remarkably expedite our participant's progression. Specifically, schemes (e.g., count-on, break-apart-make-ten) that may take typically achieving peers months and sometimes a couple of years to develop, took him only a few short weekly sessions. Remarkably, at the start of 6th grade he was still using count-all, but 13 teaching episodes later he had constructed the break-apart-make-ten scheme.

To build models of conceptual learning in terms of this process of scheme reorganization, constructivists have designed teaching experiments (Cobb & Steffe, 1983) that can promote and enable modeling of children's transition from extant schemes, to a participatory stage of a new scheme, and eventually to an anticipatory stage of that new scheme (Simon et al., 2018). One important example of the teacher's role as a facilitator is deciding when it is appropriate to move children on from solving tasks concretely to using a mental run of the same actions (Simon et al., 2018). A mental run is when the child "narrates" actions that they had previously engaged in to solve a problem. This is important because sometimes children become over-reliant on physical materials to give them an answer and do not attempt to anticipate the answer. It is this anticipation of the results that leads to the anticipatory stage of reorganizing a new scheme. Essential for this study, both physical activities and mental runs provided ample opportunities to promote the participant's reflective process through checking their answers.

We believe that our study, which drew upon a constructivist explanation of schemes, bears important similarities to two non-constructivist theories. The first, Siegler's (1987), (2006) overlapping wave theory, points out the possibility that learners alternate between strategies and concepts that an observer is likely to consider as being at different levels. He posited that learners tend to use the most efficient method they can easily and accurately access (i.e., anticipatory). We take it to suggest that a student's choice of answer-checking strategy can give a teacher important information about schemes that the student has already constructed at the anticipatory stage. We note that it does not preclude possibly also using the newest schemes available to the learner, even if yet only at the participatory stage.

The second is Vygotsky's (1978) theory, particularly the Zone of Proximal Development (ZPD) notion, suggesting children may only be able to solve tasks through interactions with more knowledgeable others (e.g., parents, siblings, teachers). Key here is that the ZPD was defined in terms of the learner's problem-solving capacities. Our study extends this distinction by applying it to the interplay between problem-solving and answer-checking schemes, while further linking, respectively, their Zone of Actual Development (ZAD) and ZPD on one hand with the anticipatory and participatory stages on the other. Tzur and Lambert (2011) postulated such a linkage, suggesting that a participatory stage is a conceptual correlate of ZPD. Because at this stage access to a scheme depends on being prompted, more knowledgeable others may provide "hints" that foster learners' shift to a higher level.

2.2. Content-specific constructs: additive schemes

Steffe and colleagues (Norton, 2018; Steffe, 1992; Steffe & Cobb, 1988) modeled specific schemes that could plausibly underlie children's observable actions and language. We focus here on schemes for additive reasoning. Each scheme draws on a child's concept of *numerosity*—the child's anticipation that, once items of a collection - perceptual, figurative, or abstract units of one (1 s) - have been counted, the number word associated with the last item will remain the same despite any changes in appearance (unless adding/-removing items). The goal in each additive scheme we focus on is to make definite the numerosity of two or more collections of items, each of which has a known numerosity (e.g., 7 steps and 5 steps are known, find the numerosity of the combined collection). Specifically, our study focused on count-all, prenumerical count-on, numerical count-on, count-up-to, count-back, and Break-Apart-Make-Ten (BAMT). Table 1 illustrates how a child would solve a task using each of these schemes.

As we noted above, the goal in those schemes is to determine the numerosity, and later the numerical value, of a combined (or decomposed) collection. We focus on the difference in the child's activity and items (units) on which they act. The first of these schemes is count-all, where children count every object in both collections of items (Murata et al., 2017). Having counted each collection separately, children then count, from one, all items in the two collections. They likely operate on perceptual or figural units of 1 (Steffe & Cobb, 1988) to make the numerosity of the sum definite, that is, the scheme's result (when using physical objects to count, this can be called perceptual count-all).

The next conceptual level-up is the count-on scheme, where the child begins counting by ones from the second addend (Tzur et al., 2021). Count-on is more advanced (and challenging) than count-all because it requires both the competence to start counting from a number other than one and the ability to consider objects as simultaneously belonging to an addend and the sum (Fuson, 1992). Children conceive of the first addend as a unit and then bring forth an activity of iterating the singletons (1 s) of the second addend while keeping track of the number of those iterations to know where to stop and thus determine the sum (result).

Within count-on, Steffe and Cobb (1988) distinguished two, qualitatively different schemes: prenumerical count-on and numerical count-on. Prenumerical count-on involves representing the entire second addend *simultaneously* and then saying the number words as they point to each item in the representation, using count-on from the first addend. They emphasized that at this point, the child can either be counting perceptually or figurally. A *figural counter* is a child who can count objects they cannot immediately perceive by representing them with other objects, such as fingers or tally marks. If the child uses count-on while acting on physical objects they are using perceptual count-on and if they count-on using representations of the objects, they are using figural count-on.

A more advanced type of count-on, which is considered numerical, involves representing the second addend *sequentially* at the same time as the child says the number words (Steffe & Cobb, 1988). It is more advanced and challenging than prenumerical count-on because the child must keep track of two counting sequences, number of unit items from the start and of how many were added, so they know when to stop. Numerical count-on entails that a child has most likely constructed a concept of number as an abstract composite unit. The child conceives of the first addend as a part of ("nested within") the total and intentionally keeps track of the second addend through double counting its items.

There are tasks, such as subtraction, that children assimilate into and can solve by count-up-to and count-back schemes (Baroody, 1984; Fuson, 1992). The count-up-to scheme involves counting forward from the subtrahend to the minuend and stating how many numbers have been counted (Baroody, 1984). There is evidence that, when the difference between the numbers is small, count-up-to is conceptually easier than the count-back scheme (Fuson, 1992). However, count-up-to does not always get used because it does not match children's conception of subtraction as a "take away" procedure (Baroody, 1984).

The count-back scheme involves counting downward from the minuend the number of 1 s equal to the subtrahend and stating the last number counted. This is more advanced than count-up-to because the child must simultaneously keep track of two different counting procedures, one that increases and one that decreases (Baroody, 1984). In both "subtraction" schemes, the child assimilates a task into a situation with both numbers as composite units, bringing forth an activity of iterating singletons forward (count-up-to) or backward (count-back) to determine the numerical difference (result). Both schemes can involve either simultaneous or sequential representations although, as we explain later, count-up-to can only be simultaneous as an answer-checking strategy, not as a problem-solving strategy.

At the next level up, with the BAMT scheme, the child assimilates a task into a numerical scheme in which the addends (e.g. 7 and 8) are each considered a unit embedded within 10, along with an additional unit (3 or 2, respectively). Critically, the task does not include 10 or 2. Rather, it is the child who calls upon that embeddedness to then use it, strategically, to decompose one addend into two smaller composite units, so it can be combined with the other addend to make ten (Tzur et al., 2021).

Murata et al. (2017) distinguished between two BAMT types. In the first one, the child makes the ten and uses count-on by ones from the ten; in the second the child makes the ten and adds the left-over part to ten. For example, to add $8 + 6$, the child considers ten as composed of $8 + 2 = 10$, sets a subgoal of obtaining that 2 by decomposing 6 into $2 + 4$, adds $8 + 2$ to make 10, not losing sight of the 4 (sub-unit of 6) which is added to the 10 to produce 14 as an answer. In this example, we emphasize that all units the child has been operating on are numerical composites, not just 1 s. We now turn to relating this content-specific part (schemes) to issues surrounding the work with diverse students.

Table 1
Schemes for additive reasoning.

Schemes	Example	Activity
No independent strategy		
Count-All	You walk 3 steps from a "Start" point and then 4 steps. How far are you from the start?	Counts all items in both groups of objects, starting from 1 (e.g., 1–2–3–4–5–6–7)
Prenumerical Count-On		Counts 2nd addend starting from end of 1st addend, e.g. uses finger gnosis to **simultaneously** raise 4 fingers and point as count each one 3; 4–5–6–7
Numerical Count-On		Counts re-presented counting acts of the 2nd addend starting from the end of the 1st addend (e.g. **sequentially** raising each finger in synch with stating the number 3; 4 (first finger) −5 (second finger) −6 (third finger) −7 (fourth finger)). Anticipates stopping when recognizes the finger pattern for 4.
Count-Up-To Simultaneous	You have 5 candies but want 8 altogether. How many more candies do you need?	Due to completing the problem using a different scheme, 3 fingers are already up; count-up from 5; 6–7–8. Recognizes finger pattern for 3.
Count-Up-To Sequential		Says 5, then raises a finger in synch with each number counted; 6 (first finger) −7 (second finger) −8 (third finger). Recognizes finger pattern for 3.
Count-Back Simultaneous	You have 8 M&Ms. You ate 3. How many are left?	Puts up 3 fingers, then points to each finger while counting backwards; 7 (first finger) −6 (second finger) −5 (third finger).
Count-Back Sequential		Puts up 8 fingers, put down a finger as count; 1 (first finger) −2 (second finger) −3 (third finger). Recognizes finger pattern for 5.
Break-Apart-Make-Ten	8 people get on a double-decker bus at stop A and then 7 people get on it at stop B. How many people are on the bus?	Using the Rekenrek. Pushes 8 beads across on bottom row. Decomposes 7 by pushing 2 more beads across on the bottom row and 5 beads on the top row. Counts 10; 11–12–13–14–15.

3. Literature on cognitively diverse students

3.1. Equity in mathematics

Equity in mathematics means that every child has the chance to learn mathematics that helps them to make meaning of the world, empowers them to make decisions (Aguirre, 2009), and engage in a "meaningful intellectual life" (Greenstein & Baglieri, 2018, p. 140). To offer appropriate opportunities for all children to engage with meaningful mathematics, we assume that all children are "capable mathematical thinkers whose thoughts are to be engaged with and cultivated" (Greenstein & Baglieri, 2018, p. 140).

Yet, much of the current research on the mathematics learning of cognitively diverse children focuses on procedural learning of basic mathematics, with few opportunities for the child to express their own ideas or engage in abstract thought (Tan et al., 2019). In Bouck and Bone's (2018) review, they detailed 11 interventions, many of which are possible to implement in a meaning-making way but are often implemented in a procedural manner (Moscardini, 2009). A frequently recommended intervention is to break tasks down into small steps so that children can work independently and thus reduce their learned helplessness (Allsopp et al., 2007). However, Greenstein and Baglieri (2018)critiqued this small-steps approach to teaching mathematics on the grounds that it makes it hard to build up the idea of a concept when it is presented one part at a time.

There are four main reasons why it is important to study children with cognitive diversity, particularly using teaching experiments and conceptual analysis, as in this study. First, if the goal of cognitive psychology is to improve our understanding of human cognition, then we need to include a much broader spectrum of people (Basnight-Brown et al., 2023), including those who are cognitively diverse and may process information differently than the majority of their peers. Second, there is a question of equity, with current mathematics courses that seem not to serve a growing number of children. The 2024 National Assessment of Educational Progress (The Nation's Report Card, 2024) showed that 24 % of 4th grade students are performing below the basic level in mathematics. The children with the lowest attainment tend to make the lowest gains in mathematics and thus the achievement gap widens (Marks, 2014), so we need to study how these children learn so that we can improve their outcomes. Third, the different ways in which children with cognitive diversity think and process information may lead to new ways of promoting learning for all children (Greenstein & Zhang, 2022). Fourth, due to the slower pace of learning of some children with disabilities it may be easier for us to pinpoint processes of cognitive change that are relevant for all children. Eriksson (2008) found that children with intellectual disabilities in his study went through the same additive schemes as typically developing children, albeit at a slower pace. The similarity of conceptual pathways between the two groups of children suggests that what researchers learn about learning from one group can be applied to the other group and vice versa.

Greenstein and Zhang (2022) consider disability as difference that is a "subjective, interactional, social construct" (p. 3) and locate the problem in the interaction between society (e.g., teachers) and the individual. From this perspective, change needs to happen within educational settings to enable the children's inclusive participation. For our work with a cognitively diverse child in this study, "inclusive" does not pertain to working with other children. Rather, it focuses on making sure each task the child is engaged in solving is inclusive of both their extant (anticipatory) schemes and the one intended for their learning. When suitable teacher-child interactions happen during a constructivist teaching experiment, children demonstrate the ability to reason mathematically (Potari, 2000). From this perspective of difference, we assume that disabilities are variations that can be accommodated with adjustments to the conceptual, social, and material resources that promote children's learning. As we shall demonstrate in our data analysis, the developmental progression of additive schemes (explained above) provides a foundation for creating strategic conceptual "inclusions" in nurturing their advances. One essential reason for this claim is that those schemes focus on what children do know and can use to move forward conceptually, that is, an asset-based perspective.

3.2. Cognitive diversity and additive reasoning

We have labeled the participant in this case study as having cognitive diversity. This is not a frequently used term in mathematics education, but we used it to reflect the lack of information we have on his actual diagnosis (due to constraints posed by our human-subject protocol). The term cognitive diversity also properly reflects what we could observe by interacting with the participant, whose cognitive style was different than other children's and this manifested itself in language delays and mathematical difficulties. Given these outward manifestations of the participant's cognitive abilities, in this section we will discuss how children with language delays and mathematical difficulties fare with additive reasoning.

Our study draws on research with children with specific language impairments that showed these children have difficulties with counting and basic calculations, but they can understand arithmetic principles (Donlan, 2007). Difficulties with counting may lead to difficulties with basic facts, as shown by Koponen et al. (2007), who found that ten-year-olds with specific language impairments still tend to use finger counting strategies to solve addition facts. The instability of their counting prevented them from moving on to more efficient strategies. This characterization is particularly relevant to our participant, whom we began working when he was still only using the count-all scheme.

Children who experience mathematics difficulties often have challenges with additive reasoning (Kim et al., 2022). Additive, numerical reasoning is the building block for other mathematical reasoning and thus it is important to consolidate such reasoning before proceeding to multiplicative reasoning (Tzur et al., 2021). We will demonstrate that, given the right level of tasks, children with mathematical difficulties can develop more sophisticated additive schemes.

It is important to note that even children with the same diagnosis of language difficulties will be affected in different ways (Marchand & Schlimm, 2025). Therefore, conclusions drawn about specific diagnoses are quite general and may not apply across the board. Neurodiversity framework accepts difference as a natural part of human development and acknowledges the strengths and challenges that come with different profiles. In our study, this implied working with the participant's available schemes as cognitive assets and promoting learning as reorganization of those within his overall linguistic and physiological abilities.

4. Methods

4.1. Context

This study was part of a larger research and professional development project, focusing on teaching mathematics conceptually (grades K-5) and funded by a small school district in the USA southwest region (see Funding). Following the university's human subjects protocol, our participants were seven of the nine 6th grade students (age 11–12), designated as having mathematical difficulties at the school, who provided parent/guardian consent and student assent.

As part of the legally required settings for students with differences, these students were in the all-inclusive classes of that school. In each of these classes, those students were supported by special education teachers, who also worked with them in an additional, one-on-one setting in a designated room. For our work with these students, the school identified a classroom on the second floor of the building that was not in use, where we covered the windows facing the school's corridor to create privacy and comfort for the students. To address the research question, we conducted a constructivist teaching experiment (Cobb & Steffe, 1983). We used a teaching experiment because it is a qualitative methodology designed to infer into and build models of learners' construction of conceptual structures and operations (i.e., schemes).

4.2. Participant

In this paper, among the seven students, we chose to focus on Mr. Happy (MH; a pseudonym the student chose for himself) for two main reasons. First, at the start of the study he showed the least advanced, prenumerical count-all scheme, coupled with quite unstable use of one-to-one correspondence between items and number words. That is, when we began working with Mr. Happy, he was yet to construct a concept of number as a composite unit.

In fact, while using that strategy, we also witnessed mistakes in counting objects (1-to-1 correspondence), which led him to count collections two or three times. This instability in one-to-one correspondence appeared to result from his coping mechanism of striving to appear competent by completing tasks as fast as possible. Once we helped him slow down, we could establish he was a consistent and competent counter of perceptual items. We thus set a goal for his learning to advance from a count-all scheme all the way to a BAMT scheme. While on the path to that scheme, we noticed and decided to focus on the anticipatory, answer-checking schemes he used and their relationships to the scheme he was working to construct at the time.

We further allude to MH's cognitive assets while operating with count-all. Revisiting our data of MH's solutions indicates to us that he has already constructed operations necessary for what will become an expedited transition to count-on, and later to BAMT. Specifically, we focus on his *uniting* operation as the core of his count-all strategy, that is, the anticipated, goal directed activity of integrating two collections into a single collection – the numerosity of which he could set as a goal to figure out. Said differently, whereas the units he could operate on at the time seemed to be perceptual (cubes, fingers), his strategy for solving addition tasks included his mental combining both collections into one.

Second, although our human subject agreement prevents sharing the diagnosis and Individualized Educational Program of MH, we were told that a major hurdle of working with him would be communication. However, we soon realized that while working on tasks within his extant schemes' reach, to explain his solution strategies or checking his answers MH could communicate with us through speaking and attentively listening to us, as well as using gestures to express his thinking. For example, when asked if he used his fingers to add two 1-digit numbers, he uttered "No" while shaking his head, waving his finger from side to side, and then pointing to his head, saying "I did it all here."

We contend that key to our productive communication with MH was the anchoring of tasks he solved in schemes that he had available (e.g., his extant uniting operation noted above). We note that, at times, he did experience difficulties in verbalizing his thinking processes, but this never deterred communication as we attuned to his actions and language.

4.3. *A constructivist teaching experiment*

A core premise of the teaching experiment methodology (Cobb & Steffe, 1983; Steffe et al., 2000) is that researchers are serving as teachers for multiple teaching episodes, linked conceptually across an extended period, to the students whose schemes they intend to model (Potari, 2000). The researcher-teachers (RTs) who worked with MH—RT1 and RT2—conducted 13 weekly teaching sessions, 25–40 min each. At any given moment, one RT interacted with MH while the other RT operated the video camera to record all teaching-learning interactions in each session and took copious field notes to guide future sessions and analysis of the video-recorded data. Due to the COVID-19 pandemic, our work with MH required all to wear masks, limiting our ability to observe his lip movements when operating silently. It also meant that RT1 did some sessions online, while RT2 was always there in person. Due to COVID-19 restrictions and other school priorities, there was a 2 ½ month gap between session four and five, which limited MH's progress during that period.

Table 2
Instructional Sequence.

Session	Instructional Goal	Range of numbers	Instructional Method	Number of prompted answer-checking incidents	Number of independent answer-checking incidents
1	Count-On; Answer-checking	Addends 4–13 Solutions 10–18	HFFS floor-tiles and game-board	2	2
2	Count-On; Answer-checking	Addends 3–11 Solutions 9–19	HFFS floor-tiles and game-board	3	1
3	Count-On; Answer-checking	Addends 4–11 Solutions 10–16	HFFS cubes and game-board	2	3
4	Count-On with word problems	Addends 3–23 Solutions 11–31	Word-problems, cubes and game-board	0	10
5	Review prior learning; Count-Up-To	Addends 3–23 Solutions 3–28	Word problems	0	0
6	Count Back	Addends 4–35 Subtrahend 8–9 Minuend 2–3 Solutions 4–43	Word problems	0	1
7	Distinguishing when to use Count-On, Count-Up-To or Count-Back	Addends 4–23 Subtrahend 15–19 Minuend 2–3 Solutions 2–36	Word problems	0	0
8	Count-On from Ten	Addends 2–11 Subtrahend 25–9 Minuend 2–4 Solutions 2–21	Word problems and Rekenrek	0	1
9	BAMT	Addends 2–34 Solutions 2–16	Word problems and Rekenrek	0	0
10	BAMT	Addends 11–16 Solutions 2–16	Word problems and Rekenrek	0	2
11	BAMT	Addends 3–9 Solutions 11–16	Word problems	0	5
12	BAMT	Addends 3–9 Solutions 3–16	Word problems	0	1
13	BAMT	Addends 7–8 Solution 15	Word problems	0	1

In the first seven teaching sessions (see Table 2), we used a two-player activity called "*How Far from the Start*" (HFFS), which RT1 designed for promoting children's advancements from count-all to count-on (see Tzur, 2019b). We illustrated it in the introduction and elaborate a bit here. The two players take a given number of steps from a starting point, either using floor-tiles or "tiles" printed on a game board. Once a number is determined (e.g., 7), Player A starts from 1, walks that number of steps, and places a sticky-note with that numeral on the tile they stopped on. Standing on that stopping point, Player B uses a second number (e.g., 5), walks that number of steps, and places a sticky-note with the numeral of that second addend on that stopping tile. A game-round ends with the players figuring out and explaining how many steps Player B has stopped from the start (e.g., 12), then checking their answer(s).

In the early sessions with MH, the game was played physically with tiles and a boardgame. However, in session 6 RT1 prompted MH to imagine playing the game and MH had to solve the problems figurally in the sense of creating unit items to count that he would impute into the "unit-less" situation (sometimes, to check the answer, they used the gameboard tiles).

To promote MH's reorganization of his count-all into a count-on scheme, RT1 used the method of hiding perceptual unit items (i.e., tiles) from MH's sight. This was done either by standing on the stopping floor tile or covering up the board space for the first addend and thus blocking MH from counting all the tiles or board spaces. Importantly, if MH was unable to figure out the total without counting-all, RT1 removed the block and allowed him to use his extant, count-all scheme. Then, we would orient MH's attention to the fact that while counting-all he (MH) actually noted and announced the first number and asked if this could help him find a different way to count (e.g., "Do you have to begin counting from 1?").

In Table 2 we provide information about the teaching episodes (aka sessions) consisting of the data for this study. From left to right, we include the session number, the intended conceptual goal for MH learning, the instructional method we used (e.g., HFFS stands for the "How Far From the Start?" game), and the number of answer-checking incidents that were initially prompted and then not prompted. We note that when playing HFFS, to provide MH with a sense of control over the task, numbers used by each player were either determined randomly (e.g., rolling dice) or purposely by RT1.

From session 4 (see Table 2), RT1 also promoted MH's construction of the intended schemes by posing realistic word problems (tasks). An example of such a task could be, "I have a bag with 9 candies. I found a bag with 5 candies. How many candies do I have altogether?" Another example could be, "You have 7 candies, but you want 12 candies altogether. How many more candies do you need?" These tasks seemed realistic for MH as they often involved food items (e.g., apples) or school materials (e.g., pencils) that he chose.

Starting in session 8 (see Table 2), RT1 changed the context of the tasks. Instead of the HFFS game, he presented tasks in the context of passengers getting on board a double-decker bus at its first and second stops. Here, the student is provided a contextual constraint: The bus has 10 seats on the lower deck that must fill up before any passengers can sit on the upper deck (also 10 seats). For example, RT posed the task: "The bus started empty in the main station. At the first stop 8 people got on the bus. At the second stop 7 more people got on the bus. How many people are on the bus altogether?" and "How many passengers are on the top floor only?"

To solve a task such as $8 + 7$, we engaged MH in using (or just thinking about) a two-rack Rekenrek model on which he pushed eight beads on the bottom rack to the left, then broke the seven into two beads pushed over to the left on the bottom rack and five beads pushed over to the left on the top rack. He knew that the bottom rack was complete with ten beads, so he initially used count-on from ten on the top rack, "11–12–13–14–15" and later just stated the sum after mentally adding $10 + 5$. At first, RT asked MH to show his actions by moving beads on the Rekenrek (see Fig. 2). However, from session 9 they asked MH to imagine and state in his own words what the Rekenrek would look like as he worked to solve the task mentally (see "mental run," Simon et al., 2018).

4.4. Data analysis

In a teaching experiment, the trustworthiness of the qualitative findings depends on the analytical process, therefore we detail it here. We used three iterations to analyze data from all fourteen sessions. First, after each session, we conducted an ongoing analysis session to discuss major events, explain their significance, and plan the next session's tasks. Second, team members read the transcripts of each session individually, highlighting segments with critical events (Powell et al., 2003) such as changes in anticipation, and teaching moves that seemed beneficial for those changes. This iteration included hypotheses of MH's reasoning, which we modeled based on his observable behaviors. Third, we discussed (while re-observing) the highlighted segments to identify compelling evidence for our inferences and conceptual claims. Finally, we organized all segments selected for the analysis in this paper into a story line that conveyed MH's *conceptual path* from count-all to BAMT.

As we noted in the introduction, it was only through several instances of the second and third analytic iterations that we realized the importance of MH's answer-checking strategies to his learning – and our understanding of it. The researchers' goals in a teaching experiment are flexible; they depend greatly on the actions of the child and the hypotheses created and tested by the RT (Potari, 2000). Due to MH's actions, the RT introduced answer-checking to MH in session 1 (see Table 2) but from the middle of session 3, to a large extent, MH took over the task of answer-checking independently. Thus, the RT's goals around answer-checking changed to a mostly observational stance, occasionally orienting MH's attention to a particular strategy he could be using.

As we noted in the introduction, answer-checking was not the primary instructional goal of the sessions. Thus, there was variability in how often MH used it and this study should be considered a pilot study on the topic of answer-checking. Once we realized the importance of answer-checking, we revisited the story line of his path and coded every task he solved also in terms of the strategy he used to check his answers, referring to the schemes of additive reasoning discussed earlier. Instances in which MH checked his answer became an answer-checking episode, and these were coded in terms of the schemes for additive reasoning. We focused our coding on whether he used the answer-checking strategies independently or was encouraged by the RT to use a particular strategy.

5. Results

In this section we present and analyze data excerpts from the ten episodes in which MH used answer-checking. Across the ten episodes there were 34 incidents of answer-checking, 27 of which MH implemented by-and-large independently (see Table 2). These episodes were not spread out evenly across the sessions because they were not the main goal of the teaching experiment. We first provide a summary of the relationship between the schemes that he independently used for answer-checking versus problem-solving and then delve into qualitative analysis of the relationship.

In our retrospective analysis, we found that MH's answer-checking schemes tended to be one conceptual step behind the ones he used to solve tasks (see Table 3). Overall, his independent answer-checking schemes were one level behind his problem-solving schemes 56 % of the time, two levels behind 22 % of the time, and at the same level 19 % of the time. We should note that when his answer-checking and task-solving schemes were at the same level, his answer-checking tended to be inaccurate (60 % inaccuracy when the levels were the same, as opposed to 14 % inaccuracy when the levels differed). As MH became more proactive in checking his answers independently, his independent answer-checking responses were more consistently one conceptual level below his task-solving response. From session 6, 73 % of the time his answer-checking strategy was one conceptual step behind his problem-solving strategy, whereas 18 % of the time it was two steps behind, and there was one instance when his answer-checking strategy was the same as his problem-solving strategy. To illustrate and further explain these findings conceptually, in the following sections, we analyze the data qualitatively.

5.1. *No independent answer-checking strategy*

When we began working with MH, he did not seem to use strategies for checking his answers, nor did he verbalize how he had found his answer. Thus, RT1 verbalized the importance of checking his answers and modelled two different ways that, RT1 inferred, MH could use to check his answers at that time: prenumerical count-on and count-all. RT1 persisted with encouraging MH to check his answers and suggesting strategies that MH could use until the end of session 1 (see Table 3). Then, MH seemed for the first time to take over the role of choosing a strategy, count-all (discussed in the next section), that he used for answer-checking. We take his development of independent checking as an indication of his confidence in using extant, readily available schemes, such as count-all (see Siegler, 2006). In turn, his answer-checking work could be assimilated back into his solution schemes, thus providing an immediate possibility for further reflection on both.

In session 2, RT1 continued to encourage MH to check his answers and suggested strategies for him to use, with MH again independently choosing his own answer-checking strategy in the last task of the session (see Table 3). In session 3, RT1 twice suggested a particular answer-checking strategy, but the other three times MH chose his own answer-checking strategy (see Table 3). From session 4, MH almost always *independently* selected and used an answer-checking strategy that seemed available to him (see Table 3).

To summarize, at first RT directed MH about the need to use answer-checking and the type of answer-checking scheme to use, so we

Table 3
Conceptual distance between problem-solving and independent answer-checking strategies across sessions.

Session	Problem-Solving Scheme	Answer-checking Scheme	Number of levels between two schemes	Accurate
1	Prenumerical Count-On	Count-All	1	No
1	Prenumerical Count-On	Prenumerical Count-on	0	No
2	Prenumerical Count-On	Count-All	1	Yes
3	Prenumerical Count-On	Count-All	1	Yes
3	Prenumerical Count-On	Count-All	1	Yes
3	Prenumerical Count-On	Count-All	1	Yes
4	Numerical Count-On	Count-All	2	Yes
4	Prenumerical Count-On	Prenumerical Count-on	0	No
4	Prenumerical Count-On	Count-All	1	Yes
4	Prenumerical Count-On	Prenumerical Count-on	0	No
4	Numerical Count-On	Count-All	2	Yes
4	Prenumerical Count-On	Prenumerical Count-on	0	Yes
4	Numerical Count-On	Count-All	2	No
4	Numerical Count-On	Prenumerical Count-on	*1*	No
4	Numerical Count-On	Count-All	2	Yes
4	Numerical Count-On	Recall	NA	Yes
6	Count-Up-To Sequential	Count-Up-To Simultaneous	1	Yes
8	Prenumerical Count-On	Prenumerical Count-on	0	Yes
10	BAMT	Numerical Count-On	1	Yes
10	BAMT	Prenumerical Count-on	2	Yes
11	BAMT	Numerical Count-On	1	Yes
11	BAMT	Numerical Count-On	1	Yes
11	BAMT	Numerical Count-On	1	Yes
11	BAMT	Numerical Count-On	1	Yes
11	BAMT	Count-Back Simultaneous	2	Yes
12	BAMT	Numerical Count-On	1	Yes
13	BAMT	Count-Up-To Sequential	1	Yes

cannot claim that these incidents reflect MH's own conceptions. It is only once MH started to use answer-checking independently that we can claim that there has been progress in MH's conceptual development of problem solving and answer-checking schemes. Next, we demonstrate MH's answer-checking with his available count-all strategy.

5.2. Count-all as a solution and an answer-check scheme

As stated above, towards the end of session 1, MH attempted to implement an answer-checking strategy (for 8 +7) on the gameboard, with no prompting (see Table 3). This took place after he had already solved an equivalent task by counting all floor tiles.

Excerpt 1: MH's use of count-all to check an answer (8 +7).

MH: [It's] Fifteen. (Begins counting from the beginning, but when he gets to the ninth space pauses for a second). No. (Begins counting again, but this time begins at the eighth spot) 8; 9–10–11–12–13-(counts this space also as)-14–15–16. No.

MH had already used count-all to find the answer to eight spaces and seven spaces on the floor tiles and thus anticipated it should also be 15 spaces on the gameboard. Yet, he seemed perturbed by his answer as indicated by his spontaneous decision to check his answer with count-all. This perturbation likely prompted his reflection on the two instances of using count-all, as indicated in his change of answer-check strategy. Thus, after he began checking using count-all (up to 9), he changed his mind about his chosen strategy and started again with a prenumerical count-on. At this point his count-on strategy did not yet seem stabilized enough for him to independently get an accurate answer, so he ended up at 16 due to an error in his execution of one-to-one correspondence. Having reached two different effects, he seemed to consider his answer-checking as wrong and followed RT1's suggestion to check his answer accurately by going back to the count-all strategy. In session 1, likely due to the gap between his first (15) and second (16) answers, we came to attribute to MH a conclusion that it was a good idea to check his answers, but he was yet to do this correctly on his own.

In session 2, MH seemed to solidify his ability to use figural items (his fingers) for a prenumerical count-on strategy while solving an addition task. For example, when Player 1 moved seven tiles and Player 2 moved four additional tiles, MH simultaneously put up four fingers and then uttered while pointing to each finger, "7; 8–9–10–11." We do not attribute to him a numerical concept at this point, as his strategy included clear use of fingers as figural items standing for the tiles with numerosities that he had established before. Then, when RT prompted him to check his answer without specifying any particular method, MH independently initiated and accurately used perceptual count-all by moving his gaze along the gameboard tiles, which is one conceptual step below his prenumerical count-on scheme. He then repeated independently and accurately using the count-all scheme for answer-checking in three out of five opportunities during session 3 and six out of nine opportunities during session 4 (see Table 3). It is these independent answer-checking responses using a count-all scheme, which indicate that MH's answer-checking scheme was one level behind the prenumerical count-on scheme he had been using to solve these tasks.

5.3. Prenumerical count-on as an answer-check scheme

Early in session 3, RT1 engaged MH in a task and MH's reasoning indicated to us that he was well into the participatory stage of prenumerical count-on. That is, with a non-specific prompt, he could regenerate that scheme to solve a problem instead of reverting to count-all. Thus, the rest of the session was dedicated to fostering his construction of a numerical count-on strategy. To recall, when using numerical count-on to add 9 + 5, MH would count-on from 9 by saying the counting sequence 9; 10, 11, 12, 13, 14 while raising a finger for each counting word in the sequence from 1 to 5, as a means to know when to stop counting (i.e., stopping once he raised his fifth and last finger; see Risley et al., 2016). At this point, we inferred that not only "9" and "5" were numerical composite units for him but perhaps also "14" as the anticipated, single number resulting from his count-on, goal-directed activity.

This inference is critical in terms of the role MH's reflections through future assimilation of answer-check processes would serve in his rather expedited transition. It resonates with the point we made about MH seemingly possessing some of the essential operations (e.g., integrating) needed to construct numerical count-on. Specifically, the integrating operation seemed readily available to him. Thus, solving a task using numerical count-on (e.g., at the participatory stage) and then checking it using prenumerical count-on and/or count-all opened the "cognitive door" for his assimilation of the latter into the former. Such assimilation, in turn, would promote gradual reorganization of the numerical count-on in the sense of linking the first and second dyad of that scheme due to extended confidence in the result being linked to the situation.

By the end of session 4, MH was independently using the numerical count-on scheme to solve tasks, with the only prompt for solving the HFFS task (18 +6) being RT1 restating the first addend:

Excerpt 2: MH's use of numerical count-on to solve a task (18 +6)

MH: 19 (puts up thumb), 20 (puts up index finger), 21 (puts up middle finger), 22 (puts up ring finger), 23 (puts up pinky), 24 (stops with thumb on other hand).

MH's solution in Excerpt 2 indicates that, with a slight prompt, he assimilated the task (18 +6) into his evolving numerical count-on scheme. Specifically, here he seemed to be using his fingers sequentially, initiating the count of each finger as an item standing for the tiles about which the task asked (tiles that were not available perceptually). When questioned, he seemed confident that he needed to stop using count-on with his fingers when he had put six of them up. Due to the very non-specific hint from RT1, we inferred that MH was well into his participatory stage in the construction of numerical count-on. We say this based on our inference that, when using his fingers to keep track, it was in tune with his anticipation of needing to know where to stop, not just mechanically using the fingers as figural items. Said differently, the fingers were (indeed) figural items that he brought forth but raising them sequentially the way he did was directed toward *his* "know-where-to-stop" goal (Tzur & Lambert, 2011).

Knowing that MH could also assimilate the same task into either a prenumerical count-on or a count-all scheme, RT1 prompted him to use the Unifix cubes (figural items for the HFFS task) to check his answer. As one would expect, MH's default strategy was count-all. However, when encouraged by RT1 to try a different way, in a direct way indicating the number to start with, MH instead used a prenumerical count-on strategy to check the answer he just obtained using numerical count-on:

Excerpt 3: MH's use of prenumerical count-on to check an answer (18 +6 =24)

RT1: You counted from one, all the way up. Is there a way for you to start from eighteen, because you already have eighteen, and only count Mr. R's?

MH: Yeah. Easy. (Goes to RT1's tower and begins to use count-on): 19–20–21–22–23–24.

Data in Excerpt 3 indicate that, with RT's specific prompt orienting MH to consider that he already counted the first addend, MH seemed highly confident ("Yeah. Easy") that he could check the answer using his (prenumerical) count-on strategy. We note that the prompt was direct and thus do not claim independent answer-checking. However, we emphasize the way RT1 used the context of the task (and HFFS actions) to promote MH's assimilation of the suggestion into a separate recognition of the given numbers in the problem. Thus, upon being prompted, MH was able to check it with the next-up strategy of prenumerical count-on. Combined, Excerpts 2 and 3 indicate that MH's solution schemes (prenumerical or numerical count-on) were at least one step ahead of his answer-checking strategy (count-all or prenumerical count-on, respectively).

The data we have analyzed thus far indicates a plausible difference in the accessibility of a scheme, between one that MH used for solving a task and another that he would use for checking his answer. When presented with a task, a child assimilates it into schemes they have constructed at the anticipatory stage and, *if prompted*, also into schemes they are constructing at the participatory stage. We conjecture that, once solving the task using strategies arising out of the latter, MH could fold-back to an answer-checking scheme that is one- or two-steps lower on the conceptual path while assimilating his own solution processes back into schemes he has already constructed at the anticipatory stage. It is in this sense that we explained how the assimilation of the answer-checking processes into the emerging scheme (here, numerical count-on) would promote an expedited process of using extant operations (e.g., integration) into the newly emerging scheme. It is also in this sense that we claim an independently used answer-checking strategy may serve as a solid indicator of where the child is conceptually, which is consistent with Vygotsky's (1986) notion of Zone of Actual Development (ZAD).

5.4. Numerical count-on as an answer-checking scheme

As stated above, in session 3 RT1 started to foster MH's construction of numerical count-on. This scheme seemed to have become anticipatory for MH from session 5, as indicated by it being his dominant, independently chosen problem-solving strategy thereafter. As noted in the theoretical framework, it may take children months, and at times 1–2 years, to construct numerical count-on as a reorganization of count-all. An expedited transition like MH's, during 4 short teaching episodes, would have to be studied further to make solid inferences as to how it came about so swiftly. Here, we can at best repeat our conjecture that he (a) already had available the necessary, integrating operation and (b) had repeatable experiences of assimilating a lower-level scheme used for answer-checking into the solution scheme. Such repeatable assimilations seemed to us supportive of his rather quick "cementing" of the first and second dyads of numerical count-on. From there on, we thus turned much attention to his use of this scheme for answer-checking.

The first time MH used numerical count-on as an answer-checking strategy was in session 4. We acknowledge that at this early phase of his construction of numerical count-on at the anticipatory stage, using this strategy also for answer-checking was heavily prompted by RT1. Yet, we also note that the direct interventions in these few instances were not intended for teaching a new mathematical scheme, but rather to encourage MH to use what we inferred he already had established as a new (for him) answer-checking tool. If he determined to use other, prior schemes to check answers, we would let him continue. Key here is that the RT never indicated to MH the correctness of his answers; instead, it was his answer-checking scheme that would provide such certainty to him.

This first time of using numerical count-on for answer checking did represent the beginning of a shift in MH's thinking about numbers as composite units. To this end, RT1 posed an HFFS task (12 +11) with a second addend larger than 10. Initially, MH thought for a whole minute, so RT1 directly prompted him to use his fingers. From MH's actions we inferred that MH intended to use both hands in a prenumerical manner. However, his count of the additional "1" (beyond 10), which he counted on the fingers on his left hand (13, 14, 15, 16, 17; then 18 while touching the thumb again) seemed to distract his thinking. To capitalize on MH's work, and perturbation with "having no finger for the extra 1," RT1 asked MH if he counted all 11 steps (the second addend). This task renegotiation is an instructional method used frequently to promote a child's reflection on the extent to which their actions and the task information correspond (see Hunt & Tzur, 2017). As RT1 anticipated, it seemed to create a perturbation for MH, noticing the gap between his goal (counting all 11 steps) and the effect of his actions (counting only 6 steps, which he could assimilate into his scheme of 6 being composed of 5 +1). Excerpt 4 shows how they proceeded, with the answer-checking not being independent.

Excerpt 4: MH's use of numerical count-on to check an answer (12 +11)

RT1: So, let's do it from the beginning. Start with 12. Count on your fingers. And make sure that you count all 11.

MH: (Counts each finger on both hands) 13, 14, 15, 16, 17; 18, 19, 20, 21, 22 (pauses for a moment, then utters quietly in anticipation of but without raising another finger); 23.

Excerpt 4 shows how RT1's renegotiation of the task, with an explicit suggestion to use fingers, led to the intended shift in MH's thinking, from prenumerical to numerical count-on. To resolve his perturbation (not having counted 11 fingers) in the direction intended by RT1, MH brought forth his evolving strategy of numerical (count-on). While this is not an example of independently using numerical count-on, RT1's prompt pointed to the use of fingers in a generic way, without specifying the sequential use of MH's fingers (see Tzur & Lambert, 2011). Because for MH one form of using fingers simultaneously (e.g. prenumerical) did not work – he seemed to assimilate that perturbing effect of his activity, coupled with the renegotiated task, into his evolving numerical count-on scheme. In this sense, Excerpt 4 illustrates a key theoretical point – how one's solution, coupled with the task the problem solver assimilated in the first place, can be assimilated into the situation part of another scheme, and thus trigger a different activity. It is this coupled assimilation (effect of problem solving with the renegotiated task), which here seemed to explain MH's learning in terms of a shift to numerical count-on, that we claim to also underlie his choice of an answer-checking strategy.

We note that a knowledgeable reader may imagine themself resolving the perturbation by putting up 10 fingers and just keeping in one's mind the need for one more finger. However, such an operation would entail that the problem solver could decompose 11 into 10 and 1, an operation that requires a concept of number as consisting of two possible smaller numbers, which is the very concept that MH was constructing. We emphasize that fostering the desired change in MH's count-on (from prenumerical to numerical) built on RT1's model of MH's scheme, in which decomposing 11 into 10 + 1 was not yet available. Accordingly, choosing 11 as a second addend could foster a plausible transition to numerical count-on (as it did). In turn, for us, it brought forth the analytic focus on how one's solution, coupled with the task, can be assimilated into another scheme that the problem solver relates to the task as a different way to solve it and, hence confirm (or disconfirm) the answer *for themselves and on their own*.

Further development of MH's use of numerical count-on for answer-checking happened from session 10, when his problem-solving strategy was BAMT. Thus, this will be discussed further in Section 5.6. We remind the reader that, from session 4 onward, all incidents of answer-checking were essentially independent (See Table 2).

5.5. *Count-up-to as an answer-checking scheme*

From session 5, we inferred that MH had constructed a concept of number as composite unit, as indicated by his independent use of numerical count-on to solve addition tasks. Thus, we began working with him to construct a count-up-to scheme. We did this because of our earlier work with MH's peers to foster their construction of a BAMT strategy. They showed us that numerical count-on might not be a sufficient conceptual basis for such an advance, which involves *decomposing of composite units*.

An example of a count-up-to task that we used to support laying such a conceptual basis is presented in Excerpt 5.

Excerpt 5: MH's use of sequential count-up-to for problem solving (28 +?=32) and simultaneous count-up-to for answer-checking.

RT1: You're at 28. I want to go from 28 to 32. How many more steps do I need to take?

MH (immediately): 28; 29 (puts up thumb), 30 (puts up index finger), 31 (puts up middle finger), 32 (puts up ring finger and looks at the results of the 4 raised fingers). Four.

(Keeping his four raised fingers up, he spontaneously and independently continues to check his answer, uttering the numbers while pointing to each of his 4 raised fingers): 29, 30, 31, 32.

Excerpt 5 shows how such tasks promoted MH's 'reversing' of his count-on goal and activity. Instead of a goal to find the sum of two addends by using numerical count-on, he brought forth double-counting for a different goal, namely, to find a composite unit that completes a given addend to the given sum. Such an inference would have been stronger had we asked for a larger difference (e.g., walk to 35). However, we believe that the critical aspect of this 'reversal' of goal and activity is telling in terms of MH's independent use of numerical count-on to accomplish a different goal. Those count-up-to tasks thus seemed like a beneficial, intermediate construction of the 'sum' (e.g., 32) as a unit composed of two smaller composite units (e.g., 28 and a unit to be determined). In turn, this could open the way to intentional use of such decomposition even when a unit is not given (e.g., to add 8 +7 by "taking 2 from 7" would require conceiving of "10," not given, as composed of 8 and 2).

MH seemed to consider the actual effect of his count ("4") not only as an anticipated result of such an activity but also as a basis for his further (answer-checking) action. He immediately, independently, and spontaneously brought forth a different, one-level-down scheme, simultaneous count-up-to, as a starting point for answer-checking. He had to know of and operate on the difference (his 4 raised fingers) between the two givens in the original task. This is another illustration of the theoretical point we made earlier, of assimilating one's solution and the original task into a different scheme, here used by MH for answer checking.

We emphasize that, here, MH checked his answer without any hint from the RTs. Rather, at that point MH seemed to 'pose' for himself a new task, which we infer could be phrased as, "Is 4 *a correct number* for the steps that RT1 needs to take from 28 to 32?" From his ability to both pose this self-initiated, follow-up task and solve it using the simultaneous count-up-to strategy, we inferred that he had constructed the underlying count-up-to scheme, at least at the participatory stage.

We link this example of MH's self-posing (answer-check) task to our references to literature about the reliance of students with IEPs in mathematics on others' approval of their answers (Brousseau & Warfield, 1999). In our analysis in the prior paragraph, the key to MH's confident way of operating seemed to be that because *he* had solved the original problem, this in turn brought forth, in *his mental system,* the possibility to also check his answer by using another, lower-level strategy (simultaneous count-up-to), which he likely had available at the anticipatory stage. The key here is that it was MH, without any prompt, who spontaneously opted to leave his four fingers up (a new situation he created for himself) and count them again by saying "29, 30, 31, 32" while pointing to each finger. We do not believe he was aware of the 'reversal' involved in doing so (compared to count-up-to), but he initiated precisely the kind of 'reversal' that, to us, would be a stepping-stone in his transition to BAMT (in the sense of decomposition as 'reversal' of composition/integration). That is, his anticipatory scheme seemed to enable him to bring forth another scheme's strategy once he obtained the answer and had his fingers raised.

Again, we note that at issue for us is not the availability of the fingers per se. Rather, it is the bringing forth of the scheme for which having his 4 fingers raised could have served as a self-prompting trigger (Tzur & Lambert, 2011). Thus, his use of the 4 raised fingers did not seem to be a part of some guess-and-check strategy. Rather, it was an anticipatory way of operating for a goal (answer-checking) that, for him at this point, seemed strongly linked to the problem-solving goal.

5.6. Using numerical count-on to check BAMT solutions

We began working with MH to foster his construction of BAMT during session 8; only 7 sessions after he barely had a count-all scheme. However, it was not until session 10 that he independently used numerical count-on as an answer-checking strategy after using BAMT to solve a task. Again, we cannot overemphasize the swift transition (3 episodes once count-up-to has been constructed) from count-on to BAMT, which seems unprecedented in the literature about cognitively diverse students. For a 6th grader who began his work with us having an unstable count-all scheme, to construct BAMT in three dedicated episodes sheds light on the plausibility of helping such students by promoting the necessary transition from count-all, through prenumerical and numerical count-on, to count-up-to, all the way to BAMT.

The data in Excerpt 6 provides further support to our claim that, aside from speed of conceptual transition, a child's scheme at the anticipatory stage provides a conceptual basis for them to use it for checking an answer:

Excerpt 6: MH's initial use of BAMT to solve a problem (9 +3) and numerical count-on to check his answer.

RT1: RT2 will put the [double-decker bus model] aside so you cannot see the "bus…" I'll tell you what she is doing, and I'll ask you a question about it and you'll tell me what you think is going on… RT2 started with a bus that is empty and [at] the first stop nine people came over. Then, [at] a second stop three more passengers [got on the bus]… How many would go on the upper deck?

MH (immediately, as if counting to himself): One; Two? (Says this answer quickly, without using his fingers, seemingly decomposing 3 into 1 +2.)

RT1: Why two?

MH: One [on] bottom, two on top…9; 10, 11, 12 (As he touches his pencil to the paper three times.)

(MH then, independently, also checked his answer): 10 (Touches his thumb with his pen), 11, no. (Restarts his answer-checking) 9, 11, no. (Restarts his answer-checking the third time) 10, 11, and 12 (Touches other fingers while saying 11 and 12).

Excerpt 6 further highlights our claim of a linkage, evidenced in the child's mental run (Simon et al., 2018) for solving the task, between a problem-solving and an answer-checking scheme. At this point, MH seemed to purposely, and mentally, decompose the second addend (3) into two smaller units (2 +1). Moreover, he did this strategically to complete a 10 (in his mind he composed a unit of 9 and one more unit of 1) and then use the "easy number pair" to find the answer by adding the remaining part (2) to it. In this sense, at least for the unit of 3, he demonstrated decomposition of it in that he could operate on its parts (1 and 2) without losing sight of both and the entire unit (3). Being able to call upon and independently decompose an addend and to think of the unit not given in the problem (10) as composed of 9 + 1, indicates a higher-level scheme than count-on, as the latter indicates the child can compose numbers but not yet decompose them (Ulrich, 2015).

With this decomposition operation, MH's cognitive capacity to use numbers as decomposable composite units, seemed to bring forth his numerical count-on strategy for answer-checking. As our theoretical point entails, he seemed to assimilate his solution (effect of BAMT strategy), coupled with the task, into his numerical scheme that includes count-on as an anticipatory scheme for solving tasks. He could thus bring numerical count-on to answer-check the task. We note that when answer-checking on his fingers, MH prompted himself to independently restart the procedure twice and was able to eventually do this correctly. This repeatable experience of reassimilating a solution process into his emerging BAMT scheme seemed, again, to promote his expedited construction of the latter at the anticipatory stage. At this point, we could say that numerical count-on was anticipatory. He similarly used numerical count-on scheme to check a problem he had solved with BAMT 4 times during session 11 and once during session 12 (see Table 3).

6. Discussion

In this paper, we focused on how a cognitive diverse learner used additive reasoning schemes with whole numbers while solving tasks and then checking his answers. We demonstrated that the answer-checking schemes used by MH, a student with an IEP in mathematics, tended to evolve in tandem with schemes he used to solve the tasks. As a pilot study, MH's case thus provides a glimpse into the interplay between these schemes, founded on the additive progression we described in Section 2.2 and the student's evolving concepts: from count-all, through prenumerical count-on, numerical count-on, count-up-to and count-back, and eventually to BAMT. Again, we stress MH's remarkable progress along this rather substantial conceptual journey, over 13 short, weekly, teaching episodes all the way to BAMT.

Our analysis provides preliminary evidence of the twofold, main hypothesis we explained theoretically and set out to test empirically: answer-checking schemes typically (a) lag one conceptual step behind the schemes the child uses to solve problems and can thus serve as an assisting indication for their conceptual understanding and (b) are assimilated back into the scheme used for solving a task and thus expedite the construction of the latter. In Section 2.1, we explained why such an interplay is expected, in that a scheme for checking an answer is most likely accessible to the child at the anticipatory stage, whereas the next-up scheme is unlikely to be viably accessible. For example, our analysis of Excerpts 2 & 3 showed this for numerical count-on (solve) vs. count-all (check), and of Excerpt 6 for MH's BAMT (solve) vs. anticipatory numerical count-on scheme (check). In the following sections, we discuss our study's theoretical and practical contributions to knowledge in the fields of mathematics and special education.

6.1. Theoretical contributions

We depict our theoretical contention by returning to the new construct, *Tandem Conceptual Progression* (TCP), presented for this pilot study in the Theoretical Framework. As stated there, this construct stresses the coupled nature of conceptually adjacent schemes. TCP thus introduces a novel stance on a student's growth along a conceptual progression, such as the one in additive reasoning in this our study.

Our case study of MH, a cognitively diverse student with an IEP in mathematics, supported and helped reveal this interplay, and as importantly – the remarkable impact it had on his expedited conceptual transitions. For example, in Excerpt 1, when MH was at the participatory stage of prenumerical count-on, he started to answer-check by using the readily available count-all scheme. Later, in Excerpt 5, when he was working at a participatory stage for sequential count-up, he immediately and spontaneously used simultaneous count-up for answer-checking.

Accordingly, we suggest that an account of a student's conceptual schemes can explain (a) both their task-solving and answer-checking schemes, and (b) how the schemes differ in terms of their availability to the learner for assimilating and thus engaging in work on tasks. Whereas this pilot study revealed the possibility of TCP, further research will be required before this idea can be generalized.

We believe that our study, which drew upon a constructivist stance, particularly on scheme theory and the participatory-anticipatory stage distinction within it, can help further link this stance with Vygotsky's (1978) and Siegler's (2006) work. Tzur and Lambert (2011) have already pointed to such a theoretical linkage, which we extend here. Specifically, we consider our proposal to be an extension of Vygotsky's (1978) seminal distinction between ZAD and ZPD. While he characterized ZAD and ZPD in terms of problems a student can or cannot solve on their own, the TCP construct stemming from our study extends this distinction by applying it to the interplay between problem-solving and answer-checking schemes. Our analysis of how answer-checking schemes seemed rooted in MH's anticipatory schemes seems to lend further cognitive support to the notion of ZAD. Similarly, our analysis of his problem-solving schemes, mostly at the participatory stage of needing prompts for completion, seems to lend cognitive support to the notion of ZPD.

Our constructivist stance helps to further explain the TCP interplay in learners' use of schemes, between the current (ZAD, anticipatory) and the next-up (ZPD, participatory) ones. Most importantly, it provides a theoretical hint for future research about the linkage between learning and development as described by Vygotsky (e.g., MH's expedited reorganization of BAMT due to assimilating his ZAD within his ZPD schemes). Key here is that a next-up strategy is evolving through the participatory stage, as a reorganization of prior, anticipatory schemes. For example, as shown in Excerpts 2 & 3, count-on emerged as a reorganization of the first item counted coupled with the keeping-track strategies he used to anticipate the stoppage of a count. As a new next-up scheme is evolving, prior anticipatory schemes are readily available to the learners, who can thus "fold back" (Pirie & Kieren, 1994) to using them, at will, for answer-checking. Our analysis of MH's answer-checking schemes, and how they remained approximately one conceptual step behind the ones he used for problem-solving, lends support to this theoretical stance.

As this stance entails, and our work with MH demonstrated, the chronicle of the TCP between MH's answer-checking and task-solving schemes have been neither linear nor entirely smooth. In the beginning, his answer-checking schemes varied between two conceptual steps behind to being on a par with his task-solving schemes. His attempts to use an answer-checking scheme at the same level as his developing task-solving scheme tended to be unsuccessful, so he mostly abandoned this strategy during the second part of the teaching experiment. However, he was much more successful when using an answer-checking scheme two levels behind his developing problem-solving scheme, and he continued to occasionally use this strategy throughout the teaching experiment. Like Siegler, (1987), (2006), we believe that he folded back to strategies rooted in schemes MH had constructed much earlier because of the sense of confidence and ownership that he, and likely many other learners with IEPs in mathematics, can experience by using schemes that have served them well for a long time.

We encouraged MH's folding back to those early schemes to support his developing sense of being the mathematical authority (Brousseau & Warfield, 1999). As our data analysis indicated, RT1 did this by emphasizing repeatedly that MH's answer-checking—not a teacher's approval—determined the correctness of his problem-solving activities. Once "checking an answer" became MH's goal, it triggered the assimilation of his own solution into prior anticipatory schemes. Furthermore, if the two answers were not the same, this caused a perturbation that was resolved by MH's consideration of the reliable scheme's answer as correct (see Excerpt 1). Gradually, as our data showed in the later sessions, MH's answer-checking strategies tended to be only one conceptual step behind his problem-solving strategies.

We note that the lack of a smooth TCP profile in MH's strategy choices is consistent with the variety of strategies younger children in a mainstream setting utilize as they develop additive reasoning (Siegler, 1987). Siegler's overlapping wave theory proposes that as children develop, they use a variety of strategies that overlap in time. With time and experience children have a propensity to use some strategies while other strategies drop out of circulation (Siegler, 2006).

These overlapping waves can be seen in our data of MH's work, as he showed with his use of answer-checking strategies, using count-all from the first through fourth session, prenumerical count-on from the first through tenth session, and numerical count-on from the tenth through twelfth session. Our observation that MH shifted from having varied numbers of conceptual steps between his answer-checking and problem-solving strategies at first to mainly one level in later sessions is consistent with Siegler's (2006) finding. As he explained, and Tzur and Lambert (2011) alluded to in terms of the participatory stage during the construction of new schemes, children's tendency to have variability in their strategies while learning a new domain seems to decrease once they have more experience within a domain. In our stance, we referred specifically to the "more experience" in answer-checking schemes being repeatedly assimilated back into the task solving schemes. Once children have a more reliable web of anticipatory schemes, they will tend to use the most efficient available scheme, which our pilot study suggests is likely to be one stage down from their current task solving scheme.

6.2. Practical importance

Two important practical implications emerge from this study. The first is that teachers can support and constantly encourage children with learning differences to check their answers. Our case study of MH showed that even cognitively diverse students who have delayed communication skills can quickly learn to rely on their own reasoning by checking their answers. In this study, the researcher-teacher started out by verbalizing the importance of checking one's answers and modelled different ways to do this. By the end of session 1 MH was already selecting his own strategies to check his answers. From session 4 he always brought forth a readily available answer-checking strategy that he could independently select. This was important because it reduced MH's reliance on others (Brousseau & Warfield, 1999), giving him the authority, confidence, and competence to not rely on the teacher but rather determine on his own whether his solutions were correct and to rework problems on finding they were incorrect. We consider our approach as a potential alternative to the typical encouragement of teachers to break tasks into manageable segments to prevent learned helplessness (Allsopp et al., 2007). As Aguirre (2009) asserted about the meaning of equity in mathematics education, every child needs to have an opportunity to learn mathematics that fosters meaning making and empowers decision making.

The second practical implication is that answer-checking strategies can serve as a formative assessment tool to make inferences about strategies that children can use independently. MH used answer-checking strategies that were at the anticipatory stage and, typically, used the most efficient strategy in the anticipatory stage that, for him, made sense to use given *his* assimilation of the problem's structure. This finding has two applications.

The first is that when a child starts to use the strategy they have been previously taught for problem-solving as a tool for answer-checking, it can serve as an indication to the teacher that this strategy is now at the anticipatory stage. It is thus sensible for the teacher to move the child on to the next stage in the learning trajectory (Tzur, 2019a). In our work with MH, RT1 continually used such an indication, which was a major reason why we began paying attention to this pedagogical feature and set out to study it. Most importantly, we explained that his expedited transition, from a count-all scheme all the way to BAMT, seemed highly supported by such a focus of attention from the teacher.

The second pertains to a child who seems to routinely use an answer-checking strategy that is several levels below the level at which they are being taught. In such a case, it suggests that the intermediate steps in solving current tasks may still be at the participatory stage. Thus, the teacher is provided with a formative assessment indicator to revisit these prior levels and solidify those prerequisite schemes. Using answer-checking as formative assessment seems to untangle a dilemma posed by Tzur (2007), who pointed to the importance of assessing children so that the teacher knows how to pitch the lessons, while it is equally important not to take up too much lesson time with assessment.

6.3. Limitations and implications for future research

Our case study of MH only looked at the answer-checking schemes of one cognitively diverse student. At first, MH tended to use an answer-checking scheme that varied between two to zero levels below his problem-solving strategy, but later in the study he tended to use strategies one conceptual level below his problem-solving level. One plausible reason for this phenomenon is that he uses schemes that are at an anticipatory stage for answer-checking, but there are other plausible reasons, such as his interpretation of a request for answer-checking as an indication of the need to use a different method. Future studies can look at the answer-checking schemes of larger groups of students with diverse needs and attainment levels to increase our confidence that, for answer-checking, most children will be likely to use schemes that are anticipatory. Such studies will provide further evidence (or counter) to the TCP construct we have introduced in this paper.

Likewise, it would be important to investigate the relationship between the children's conceptual levels for answer-checking and task-solving to discover whether MH's TCP pattern was typical. Given the international and cross-disciplinary facets of this article's author team, another fruitful area for future research would be to investigate whether cognitively diverse children in other countries also utilize their anticipatory schemes when answer-checking and what relationship can be identified between their answer-checking and problem-solving strategies.

When working individually with MH, this team of researchers was able to observe and encourage his answer-checking schemes. Many teachers, however, may not have the luxury of working individually with children. Even interventions for the 2–5 % of children with the most difficulties in mathematics tend to be in small groups of two or three children (Richards et al., 2007). For this method of formative assessment to have practical significance, teachers need to learn how to analyze the answer-checking schemes of children working in a small group and then utilize this information to inform their choices of what and how to teach the next step. Future research can thus shed light on how professional development efforts for teachers may support them in taking on this idea of using answer-checking as a form of formative assessment and utilize it to inform their instruction in small intervention groups.

While it was important that MH learned to take responsibility to check his answers, what is not apparent from this study is whether he transferred this skill to other contexts. Future research could investigate how much and in what ways children transfer skills learned in one academic setting where the teacher has repeatedly urged them to check their answers, into another setting where the teacher may not be so insistent on the child checking their answers.

While our research team worked with MH, it was impossible not to note the tremendous growth in his confidence and positive affect as the intervention progressed and he made more choices about his learning and answer-checking strategies. At this point, this is anecdotal evidence and was a by-product of the main research focus. However, future studies could intentionally investigate the emotional impact of teaching children to check their own and other people's (e.g., teachers, peers) answers.

7. Concluding remarks

Encouraging cognitively diverse students to independently check their answers seems to be a beneficial practice, as well as a topic for future research, because it provides ample opportunities for carrying out formative assessment by showing us what the learner's anticipatory scheme(s) may be. This strategic topic also seems to be emancipatory, in that it allows the students to take control and ownership of their own learning. We also noted theoretical implications to this practice: it lends support to the possible linkages between the participatory-anticipatory stage distinction (Tzur & Simon, 2004), Vygotsky's (1978) core notions of ZPD and ZAD, and waves of overlapping strategies (Siegler, 2006), such as those that MH's case seemed to present for answer-checking and thus demonstrating that his conceptual development advanced through similar schemes and stages as younger children in mainstream education.

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CRediT authorship contribution statement

Cody Harrington: Writing – review & editing, Project administration, Investigation, Formal analysis, Data curation. **Helen Thouless:** Writing – review & editing, Writing – original draft, Formal analysis. **Alan Davis:** Writing – review & editing. **Ron Tzur:** Writing – review & editing, Methodology, Investigation, Funding acquisition, Formal analysis, Conceptualization. **Dagli Beyza:** Writing – review & editing, Data curation.

Declaration of Competing Interest

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Data availability

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